Math 821, Spring 2013, Lecture 15

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1 Partitions

Definition 1. *A partition of n is $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k)$ such that $\lambda_1 + \ldots + \lambda_k = n$, where*

- *n is the size of λ*
- *k is the number of parts*

Write $k(\lambda)$ for the number of parts of λ.

We've already discussed Ferrers diagrams, as well as Durfee squares and the conjugate position where $\widetilde{\lambda}$ is the conjugate of λ:

Example 1. $\lambda = (6, 5, 5, 3, 2, 2, 1)$

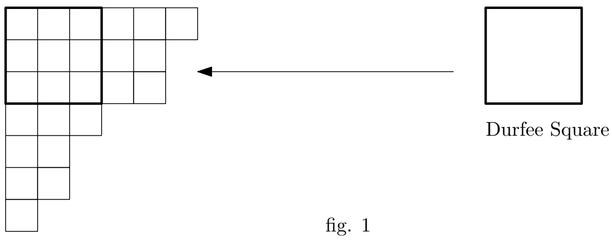

fig. 1

Definition 2. *Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k)$ be a partition. Let F be its Ferrers diagram viewed with the top left corner's coordinates at $(0, 0)$. Let $\widetilde{F}$ be F after reflecting in $y = -x$. Then the partition $\widetilde{\lambda}$ corresponding to $\widetilde{P}$ is the conjugate of λ*

Example 2. *With the previous example for λ, $\widetilde{\lambda} = (7, 6, 4, 3, 3, 1)$.*

Some specifications for partitions:

$$\mathcal{P} = \prod_{j=1}^{\infty} Seq(\mathcal{Z}^j)$$

$$\mathcal{P} = MSet(\mathcal{I}) \qquad \mathcal{I} = Seq_{\geq 1}(\mathcal{Z})$$

$$\mathcal{P} = \sum_{k=1}^{\infty} \left(\mathcal{Z}^{k^2}\right) \times \left(\prod_{j=1}^{\infty} Seq(\mathcal{Z}^j)\right)^2$$

2 Identities and Bijections

A classic problem you may have seen before:

Proposition 1. *The number of partitions of size n with distinct parts equals the number of partitions of size n with odd parts.*

Proof. Let $\mathcal{D}$ be the combinatorial class of partitions with distinct parts. Let $\mathcal{P}_o$ be the combinatorial class of partitions with odd parts.

$$\mathcal{D} = \prod_{j=1}^{\infty} (\mathcal{E} + \mathcal{Z}^j) \qquad D(x) = \prod_{j=1}^{\infty} (1 + x^j)$$

$$\mathcal{P}_o = \prod_{j=1}^{\infty} Seq(\mathcal{Z}^{2j-1}) \qquad P_o(x) = \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j-1}}$$

$$D(x) = \prod_{j=1}^{\infty} (1 + x^j) \prod_{j=1}^{\infty} \frac{1 - x^j}{1 - x^j} = \prod_{j=1}^{\infty} \frac{1 - x^{2j}}{1 - x^j} = \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j-1}} = P_o(x)$$

□

Proposition 2 (Euler's pentagonal number theorem)**.**

$$\prod_{j=1}^{\infty} (1 - x^j) = \sum_{h=-\infty}^{\infty} (-1)^h x^{\frac{h(3h-1)}{2}}$$

Before we prove this, what does it have to do with pentagonal numbers? Pentagonal numbers summarize the "size" of pentagons:

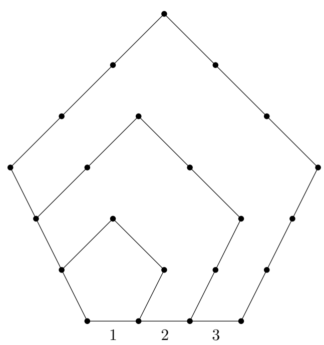

fig. 2

To simplify summing, here's a trick:

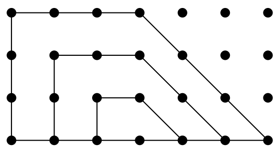

fig. 3

We want a formula for these numbers. Pentagonal numbers are given by the formula

$$h^2 + \frac{h(h-1)}{2} = \frac{3h^2 - h}{2}$$

which appears as the power of x.

Proof. First rewrite the result:

$$\prod_{j=1}^{\infty}(1-x^j) = 1 + \sum_{h=1}^{\infty}(-1)^h \left(x^{\frac{h(3h-1)}{2}} + x^{\frac{h(3h+1)}{2}} \right)$$

Now let's interpret these combinatorially. We'll use $\mathcal{D}$ again. Let $D(x, y)$ be the bivariate generating function for $\mathcal{D}$.

$$D(x, y) = \sum_{\lambda \in \mathcal{D}} x^{|\lambda|} y^{k(\lambda)}$$

Then

$$D(x, y) = \prod_{j=1}^{\infty}(1 + x^j y)$$

So $D(x, -1)$ is the left hand side of the proposition.

Furthermore,

$$D(x, -1) = \sum_{n=1}^{\infty}(d_{e,n} - d_{o,n})x^n$$

where

- $d_{e,n}$ is the number of elements of $\mathcal{D}_n$ with an even number of parts
- $d_{o,n}$ is the number of elements of $\mathcal{D}_n$ with an odd number of parts

Now let's define some more parameters for partitions. $x(\lambda) = \lambda_{k(\lambda)} =$ the smallest part corresponds to the boxes in the last row of the Ferrers diagram.

Mark them on the Ferrers diagram with an x.

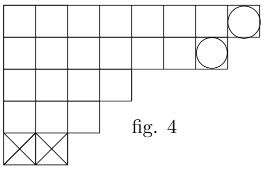

fig. 4

$O(\lambda) = \max\{c : \lambda_1 + 1 - c = \lambda_c\}$, which is the number of boxes in the rightmost reverse diagonal of the Ferrers diagram.

Now we want to build an involution $\psi : \mathcal{D} \to \mathcal{D}$ with the following properties:

- $|\psi(\lambda)| = |\lambda| \quad \forall \lambda \in \mathcal{D}$
- either $\psi(\lambda) = \lambda$ or $|k(\lambda) - k(\psi(\lambda))| = 1 \quad \forall \lambda \in \mathcal{D}$
- $\psi(\psi(\lambda)) = \lambda \quad \forall \lambda \in \mathcal{D}$

Assuming we have such a ψ, let

$$f_{e,n} = |\{\lambda \in \mathcal{D}_n : k(\lambda)\ even,\ \psi(\lambda) = \lambda\}|$$

$$f_{o,n} = |\{\lambda \in \mathcal{D}_n : k(\lambda)\ odd,\ \psi(\lambda) = \lambda\}|$$

Then

$$D(x, -1) = \sum_{n=1}^{\infty} (d_{e,n} - d_{o,n}) x^n$$

$$= \sum_{n=1}^{\infty} (f_{e,n} - f_{o,n}) x^n + \sum_{\lambda \in \mathcal{D},\ |k(\lambda) - k(\psi(\lambda))| = 1} x^{\lambda} (-1)^{k(\lambda)}$$

but applying ψ to that equation we get

$$D(x, -1) = \sum_{n=1}^{\infty} (f_{e,n} - f_{o,n}) x^n + \sum_{\lambda \in \mathcal{D},\ |k(\lambda) - k(\psi(\lambda))| = 1} x^{\psi(\lambda)} (-1)^{k(\psi(\lambda))}$$

Then by ψ, the last term of the second equation is the negative of the last term of the first equation. Thus, they are each zero.

Next, build ψ: take $\lambda \in \mathcal{D}$

- **(a)** If $x(\lambda) \geq o(\lambda) + 2$, then remove the o boxes, including any box with both x and o. Append them as a new row. The result is the Ferrers diagram of a partition with distinct parts, since after removing the o boxes, we still have at least $o(\lambda) + 1$ boxes in the last row, so we can add the o's as a new row remaining distinct.

- **(b)** If $x(\lambda) = o(\lambda) + 1$ and there is no box with both x and o, then do as in part **(a)** and we still get a partition with distinct parts.

- **(c)** If $x(\lambda) = o(\lambda) + 1$ and there is a box with both x and o, then $\psi(\lambda) = \lambda$.

- **(d)** If $o(\lambda) > x(\lambda)$, remove the x boxes including any x,o box and append one to the end of the first $x(\lambda)$ rows. This gives a partition with distinct parts and furthermore each x is appended immediately following an o.

- **(e)** If $o(\lambda) = x(\lambda)$ and there's no box with both x and o, then do as in **(d)** and get the same result.

- **(f)** If $o(\lambda) = x(\lambda)$ and there is an x,o box, then $\psi(\lambda) = \lambda$.

Now we need to check ψ has the properties it's supposed to have. As we constructed it, we checked $\psi : \mathcal{D} \to \mathcal{D}$. Also, $|\psi(\lambda)| = |\lambda|$ because we only move boxes, never add or remove them.

- In **(c)** and **(f)**, $\psi(\lambda) = \lambda$.

- In **(a)** and **(b)**, $k(\lambda) = k(\psi(\lambda)) - 1$

- In **(d)** and **(e)**, $k(\lambda) = k(\psi(\lambda)) + 1$

giving the second property of ψ.

If ψ is in **(c)** or **(f)**, then so is ψ, so $\psi(\psi(\lambda)) = \lambda$.

If λ is in **(a)** or **(b)**,

$$x(\psi(\lambda)) = o(\lambda)$$

$$o(\psi(\lambda)) \geq o(\lambda)$$

if equal, there's no x,o box so we're in the case **(d)** or **(e)** for $\psi(\lambda)$. So $\psi(\psi(\lambda))$ just places the boxes back.

Similarly, if λ is in **(d)** or **(e)**,

$$x(\psi(\lambda)) > x(\lambda)$$

(strictly greater due to distinct parts.)

$$o(\psi(\lambda)) = x(\lambda)$$

if $x(\psi(\lambda)) = x(\lambda) + 1$, then we didn't add an extra box to this row, so no x,o box in $\psi(\lambda)$. Thus we're in **(a)** or **(b)**, and so $\psi(\psi(\lambda))$ places the boxes back in their original places.

So

$$D(x,-1) = \sum_{\lambda \in \mathcal{D}, \psi(\lambda)=\lambda} x^{|\lambda|}(-1)^{k(\lambda)} = \prod_{j=1}^{\infty}(1-x^j)$$

So what are the fixed points of ψ? They must have an x,o box and either $o(\lambda) = x(\lambda)$ or $x(\lambda) = o(\lambda) + 1$.

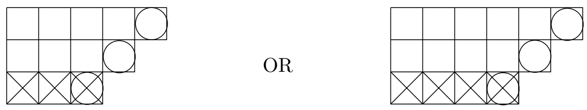

fig. 5

So the size must be $k^2 + \frac{k(k-1)}{2} = \frac{k(3k-1)}{2}$ or $k^2 + \frac{k(k-1)}{2} = \frac{k(3k+1)}{2}$ except if $k = 0$, then there is just $\mathcal{E}$ contributing a 1. So we get the desired formula. $\square$

3 References

Yeats' personal notes.